Affine Growth Diagrams

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Algebra: Rep Theory Notation

The **dominant weights** of GL_m, denoted Λ_+, are nonincreasing sequences of integers λ of length m. The dual of a weight λ^* is given by negating and reversing.

$$\lambda = (3,1,1,0,-2),\ \lambda^* = (2,0,-1,-1,-3)$$

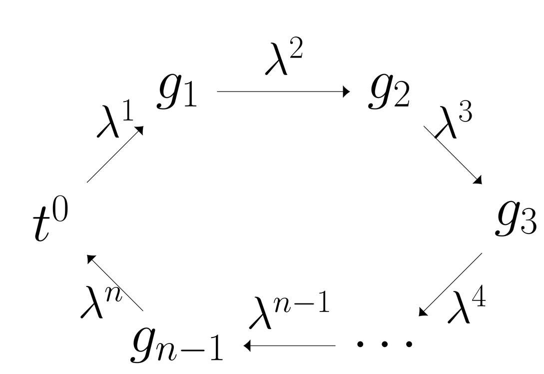
The **minuscule weights** are the fundamental weights ω_i and the dual fundamental weights ω_i^*.

$$\omega_i = (1,1,\dots,1,0,0,\dots,0),\ \omega_i^* = (0,0,\dots,0,-1,-1,\dots,-1)$$

Let $c_{\lambda^1,\lambda^2,\dots,\lambda^n} = \dim(V_{\lambda^1}\otimes\cdots\otimes V_{\lambda^n})^{GL_m}$ be the **Littlewood-Richardson coefficient**. Recall that tensoring with a minuscule rep is **multiplicity free**.

Geometry: Polygon Spaces

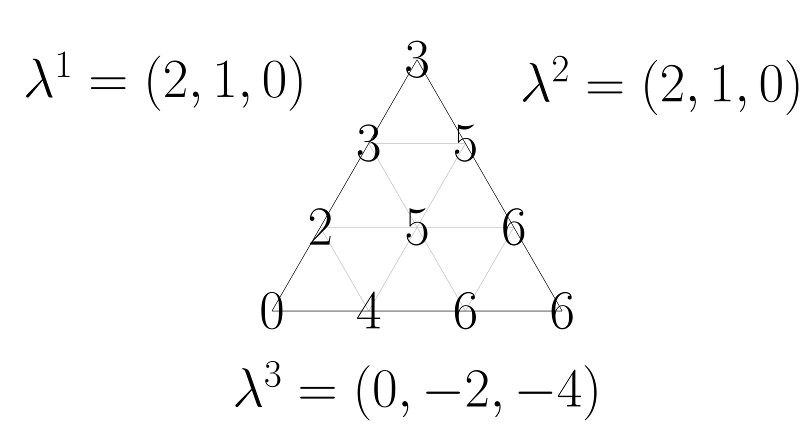
Let $Gr = GL_m(\mathbb{C}((t)))/GL_m(\mathbb{C}[[t]])$ be the **affine Grassmannian**. There is a **distance function** $d : Gr \times Gr \to \Lambda_+$. For a sequence of n dominant weights $\vec{\lambda}$ define the based **polygon space** $\mathrm{Poly}(\vec{\lambda})$ to be the set of $(g_1,\dots,g_{n-1}) \in Gr^{n-1}$ satisfying $d(g_{i-1},g_i) = \lambda^i$.



Theorem(Geometric Satake): The number of components of $\mathrm{Poly}(\vec{\lambda})$ is $c_{\vec{\lambda}}$.

Combinatorics: Hives

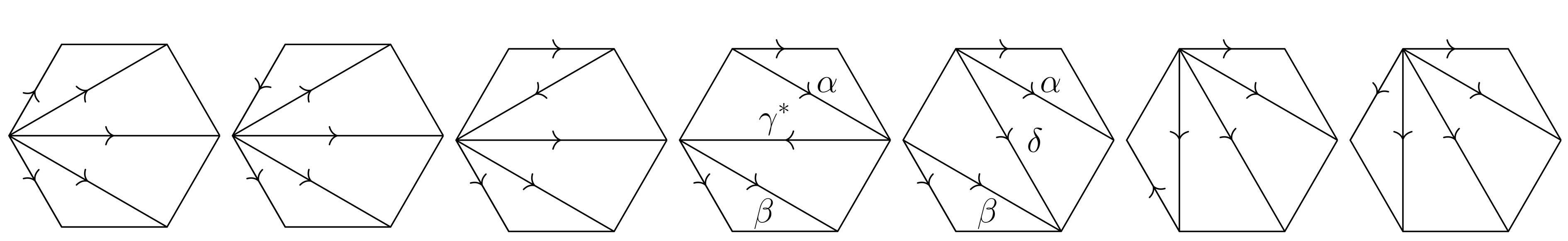
A **3-hive** is a triangular array of integers satisfying the **rhombus inequalities**, for a unit rhombus the sum across the short edge is at least the sum across the long edge. An **n-hive** is an $(n-1)$-simplex worth of integers satisfying the rhombus inequalities on each 2-face and the **octahedron recurrence**, $e' = \max(a+c, b+d) - e$.



Theorem([2]): For a sequence of n dominant weights $\vec{\lambda}$ the number of n-hives with boundary $\vec{\lambda}$ is $c_{\vec{\lambda}}$.

Extroverted Triangulations

An **extroverted triangulation** of the n-gon consists of triangles each containing an edge of the n-gon. When the λ^i are minuscule the weights along the edges of an extroverted triangulation uniquely determine an n-hive. We can get the rest of the hive labels via repeated "excavations" of 4-hives.



Main Result

Define an **affine growth diagram** to be an infinite periodic staircase labelled by dominant weights such that each unit square satisfies the local rule and the southwest and northeast diagonals are labelled by $\emptyset = (0,\dots,0)$. An extroverted triangulation defines a path in the staircase.

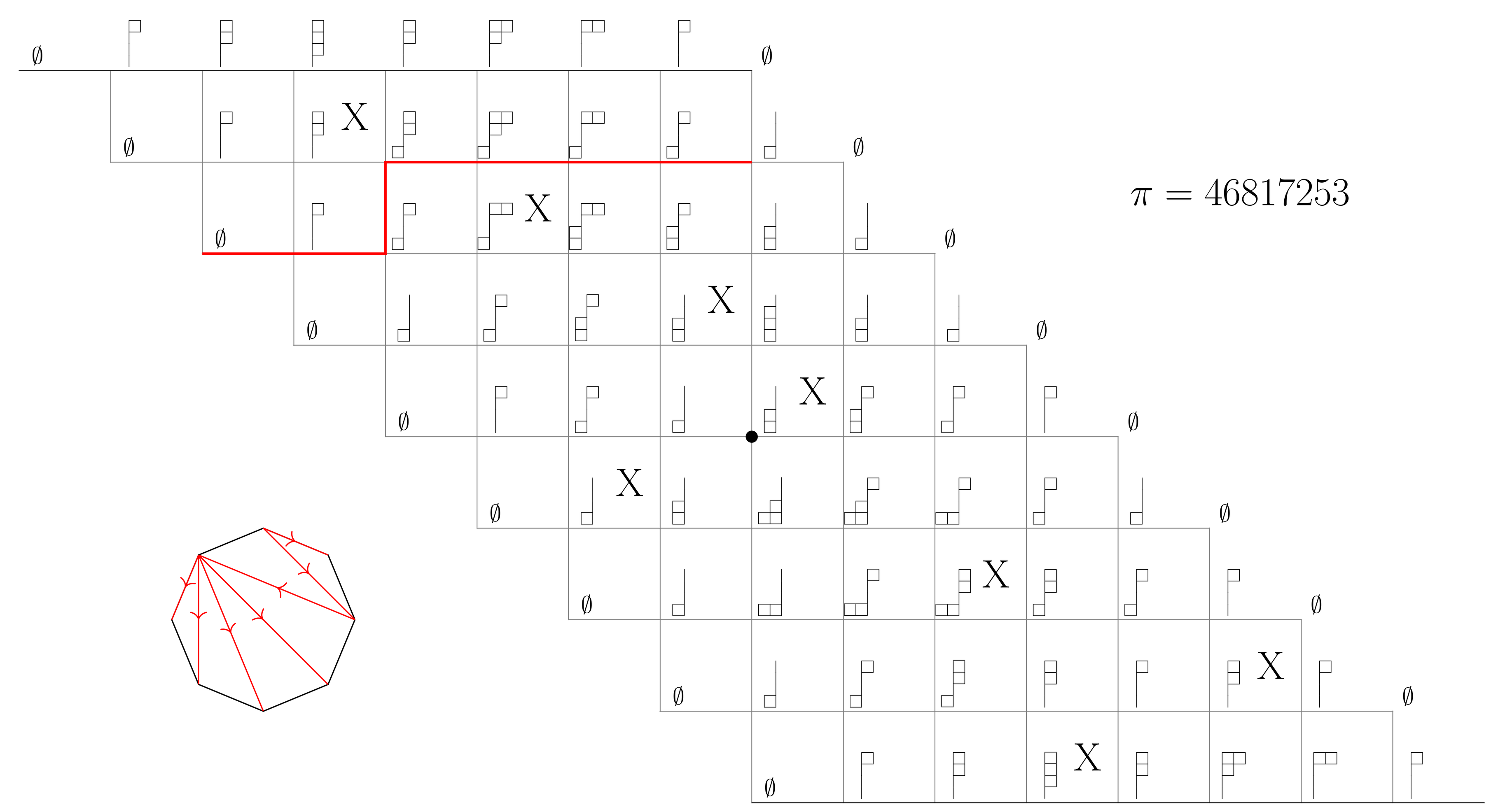
$$\begin{array}{cc} \gamma & \beta \\ \alpha & \delta \end{array}$$

The Local Rule: $\delta = \mathrm{sort}(\alpha+\beta-\gamma)$

Fix a component of $\mathrm{Poly}(\vec{\lambda})$ and the corresponding n-hive with boundary $\vec{\lambda}$. Given any extroverted triangulation τ and weights along the edges of τ, fill in a staircase diagram to the southeast according to the local rule to produce an affine growth diagram. Then the resulting vertex labels are the weights of the 1-skeleton of the n-hive. Moreover, the components of $\mathrm{Poly}(\vec{\lambda})$ are in bijection with affine growth diagrams with $\vec{\lambda}$ down the first southwest diagonal.

Arising from a different combinatorial consideration, the local rule appears in [3].

Example

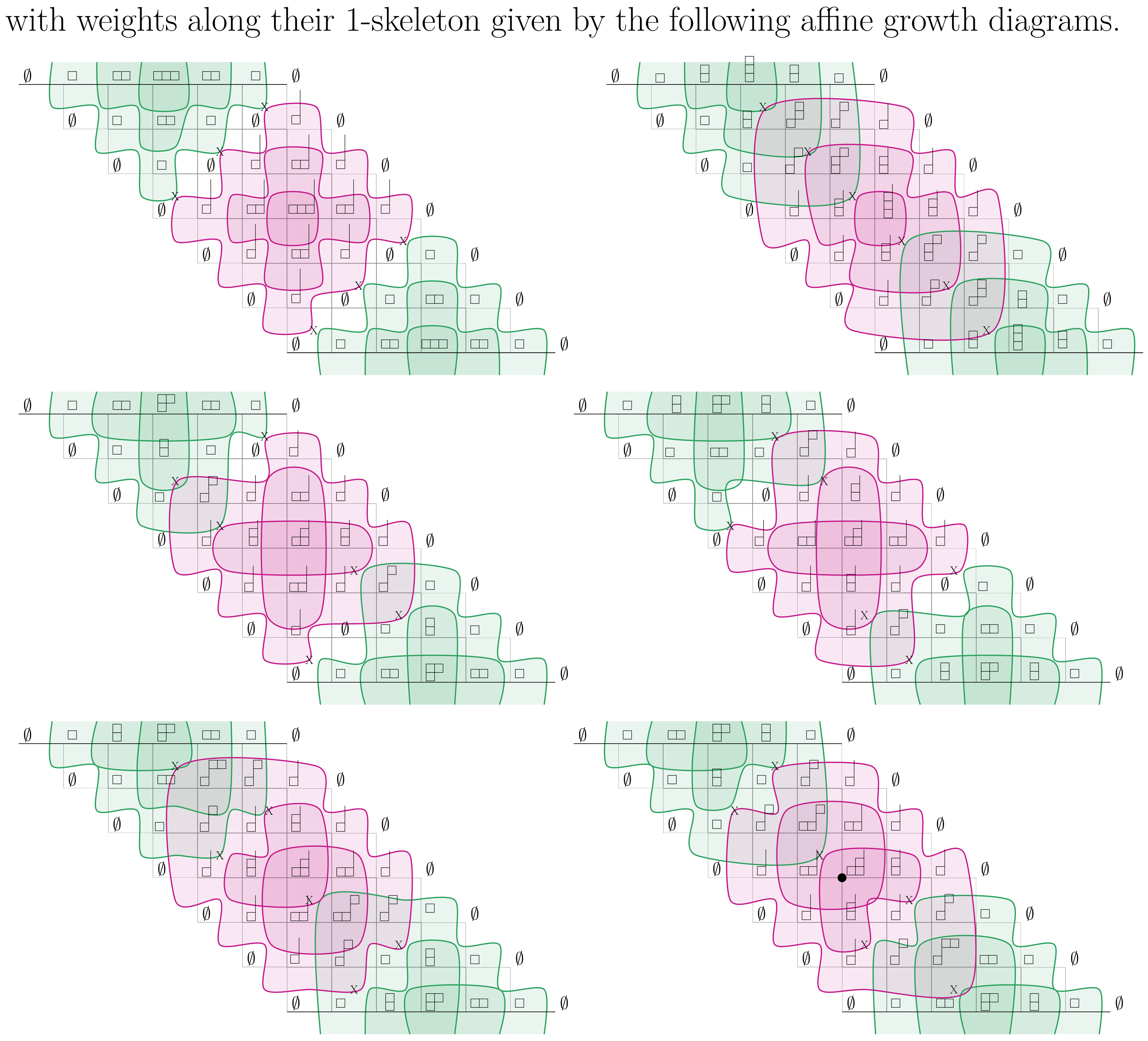


Some Takeaways

1. Rediscover Stanley-Sundaram/Roby bijection: oscillating tableaux $\leftrightarrow$ fixed-point-free involutions
2. Within bijection 1 rediscover Fomin diagrams and the RS correspondence: pairs of same shape tableaux $\leftrightarrow$ permutations
3. Can "Knuthify" 1 to get the bijection: semistandard oscillating tableaux $\leftrightarrow$ fixed-point-free $\mathbb{N}$ involutions
4. Within bijection 3 rediscover the RSK correspondence: pairs of same shape semistandard tableaux $\leftrightarrow$ $\mathbb{N}$ matrices

Full Example

Let $n = 6$ and $\lambda^1 = \lambda^2 = \lambda^3 = (1,0,0,\dots,0)$, $\lambda^4 = \lambda^5 = \lambda^6 = (0,\dots,0,0,-1)$. Then $\mathrm{Poly}(\vec{\lambda})$ has six components corresponding to six hives with boundary $\vec{\lambda}$ with weights along their 1-skeleton given by the following affine growth diagrams.



References

[1] Bruce Fontaine, Joel Kamnitzer, *Cyclic Sieving, Rotation, and Geometric Representation Theory*, Selecta Math., 20 no. 2 (2014) 609-625; math.RT/arXiv:1212.1314.

[2] Allen Knutson, Terence Tao, Christopher Woodward, *A Positive Proof of the Littlewood-Richardson Rule using the Octahedron Recurrence*, Electronic Journal of Combinatorics, vol. 11, issue 1 (2004).

[3] Marc A. A. van Leeuwen. *An Analogue of Jeu de Taquin for Littelmann's Crystal Paths*, Sém. Lothar. Combin., 41: Art. B41b, 23 pp. 1998.

[4] Tom Roby, *Applications and Extensions of Fomin's Generalization of the Robinson-Schensted Correspondence to Differential Posets*, Ph.D. Thesis, Massachusetts Institute of Technology, 1991.